Run UUID: 0c01dc3f-6281-4cbc-bcc9-ba1396fc675e
Time taken: 291s
Model: Qwen/Qwen3.5-397B-A17B
Peer Review: PASSED

IPhO 2026 - Problem T3

Part A: Work in the Pm-T

T3-A1 [0.2 pts]

Derivation & Working:

For a toroidal solenoid, we apply Ampère's law. The magnetic field $\vec{H}$ circulates along the toroid. For a circular path of radius R (the mean radius):

$$\oint_C \vec{H} \cdot d\vec{l} = H \cdot 2\pi R = NI$$

The volume of the torus is $V = 2\pi R \cdot A$, where A is the cross-sectional area. Therefore:

$$2\pi R = \frac{V}{A}$$

Substituting into Ampère's law:

$$H \cdot \frac{V}{A} = NI$$

Final Analytical Expression:

$$H = \frac{NIA}{V}$$

T3-A2 [0.6 pts]

Derivation & Working:

The work done by the external voltage source (EMF) to change the magnetic flux is given by:

$$dW_{\text{emf}} = \mathcal{E} I dt$$

From Faraday's law, $\mathcal{E} = -\frac{d\Phi}{dt}$, but since we're calculating work done BY the source (not against it), we use:

$$dW_{\text{emf}} = I d\Phi$$

The magnetic flux through the toroid is $\Phi = N \cdot B \cdot A$ (N turns, each with area A). Thus:

$$d\Phi = NA dB$$

From T3-A1, $I = \frac{HV}{NA}$. Substituting:

$$dW_{\text{emf}} = \frac{HV}{NA} \cdot NA dB = VH dB$$

Final Analytical Expression:

$$dW_{\text{emf}} = VH dB$$

T3-A3 [0.2 pts]
Derivation & Working:

From the constitutive relation for paramagnetic materials:

$$\vec{B} = \mu_0 \vec{H} + \mu_0 \overrightarrow{M}$$

Taking the differential:

$$dB = \mu_0 dH + \mu_0 dM$$

Substituting into the expression for dW_{emf}:

$$dW_{\text{emf}} = VH(\mu_0 dH + \mu_0 dM) = \mu_0 VHdH + \mu_0 VHdM$$

The first term $\mu_0 VHdH$ represents the work to change the field in vacuum (dW_{vac}), and the second term represents the work done on the paramagnetic material (dW).

Final Analytical Expression:

$$dW = \mu_0 VHdM$$

Part B: Heat and Temperature in the Pm-T

T3-B1 [1.5 pts]
Derivation & Working:

For an isothermal process at temperature T, we use the first law of thermodynamics:

$$dU = dQ + dW$$

From the table, $dU = C_M dT$. At constant temperature, $dT = 0$, so $dU = 0$. Therefore:

$$dQ = -dW = -\mu_0 VHdM$$

From the equation of state for the paramagnetic torus:

$$TMV = nKH \Longrightarrow M = \frac{nKH}{TV}$$

At constant T and V:

$$dM = \frac{nK}{TV} dH$$

Substituting:

$$dQ = -\mu_0 VH \cdot \frac{nK}{TV} dH = -\frac{\mu_0 nKH}{T} dH$$

Integrating from H_i to H_f:

$$Q = \int_{H_i}^{H_f} -\frac{\mu_0 nKH}{T} dH = -\frac{\mu_0 nK}{T} \int_{H_i}^{H_f} HdH = -\frac{\mu_0 nK}{T} \cdot \frac{H_f^2 - H_i^2}{2}$$

Final Analytical Expression:

$$Q = \frac{\mu_0 nK(H_i^2 - H_f^2)}{2T}$$

T3-B2 [1.5 pts]
Derivation & Working:

For an adiabatic process, $dQ = 0$, so from the first law:

$$dU = dW = \mu_0 V H dM$$

Also from the table, $dU = C_M dT = \frac{n\lambda}{T^2} dT$.

From the equation of state $M = \frac{nKH}{TV}$, taking the total differential at constant V:

$$dM = \frac{nK}{TV} dH - \frac{nKH}{T^2 V} dT$$

Substituting into $dU = \mu_0 V H dM$:

$$dU = \mu_0 V H \left(\frac{nK}{TV} dH - \frac{nKH}{T^2 V} dT \right) = \frac{\mu_0 nKH}{T} dH - \frac{\mu_0 nKH^2}{T^2} dT$$

Equating the two expressions for dU:

$$\frac{n\lambda}{T^2} dT = \frac{\mu_0 nKH}{T} dH - \frac{\mu_0 nKH^2}{T^2} dT$$

Dividing by n and rearranging:

$$\frac{\lambda}{T^2} dT + \frac{\mu_0 KH^2}{T^2} dT = \frac{\mu_0 KH}{T} dH$$

$$\frac{\lambda + \mu_0 KH^2}{T^2} dT = \frac{\mu_0 KH}{T} dH$$

$$\frac{dT}{T} = \frac{\mu_0 KH}{\lambda + \mu_0 KH^2} dH$$

Integrating from (T_i, H_i) to (T_f, H_f):

$$\int_{T_i}^{T_f} \frac{dT}{T} = \int_{H_i}^{H_f} \frac{\mu_0 KH}{\lambda + \mu_0 KH^2} dH$$

$$\ln\left(\frac{T_f}{T_i} \right) = \frac{1}{2} \ln\left(\frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2} \right)$$

$$\frac{T_f}{T_i} = \sqrt{\frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}}$$

$$T_f = T_i \sqrt{\frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}}$$

Final Analytical Expression:

$$\Delta T = T_i \left(\sqrt{\frac{\lambda + \mu_0 KH_f^2}{\lambda + \mu_0 KH_i^2}} - 1 \right)$$

Part C: Carnot Refrigerator of the Pm-T

T3-C1 [0.2 pts]
Analysis:

From the H vs. T diagram (Figure 3b):

- Points 1 and 4 are at the higher temperature (hot reservoir): T_h
- Points 2 and 3 are at the lower temperature (cold reservoir): T_c
- Process $2 \to 3$ is isothermal at T_c: heat Q_c is absorbed from the cold reservoir
- Process $4 \to 1$ is isothermal at T_h: heat Q_h is rejected to the hot reservoir
- Processes $1 \to 2$ and $3 \to 4$ are adiabatic

Final Answer:

"**T_h is at the right vertical line (points 1 and 4); T_c is at the left vertical line (points 2 and 3). Q_c occurs during process $2 \to 3$ Q_h occurs during process $4 \to 1$.**"

T3-C2 [1.5 pts]
Derivation & Working:

For a Carnot cycle, we use the equation of state and the adiabatic conditions.

From the equation of state $M = \frac{nKH}{TV}$:

- At T_h: $M_1 = \frac{nKH_1}{T_hV}$ and $M_4 = \frac{nKH_4}{T_hV}$
- At T_c: $M_2 = \frac{nKH_2}{T_cV}$ and $M_3 = \frac{nKH_3}{T_cV}$

From the adiabatic condition derived in T3-B2, for process $1 \to 2$:

$$\frac{T_h^2}{\lambda + \mu_0 K H_1^2} = \frac{T_c^2}{\lambda + \mu_0 K H_2^2}$$

For process $3 \to 4$:

$$\frac{T_c^2}{\lambda + \mu_0 K H_3^2} = \frac{T_h^2}{\lambda + \mu_0 K H_4^2}$$

Equating the expressions for T_h^2/T_c^2:

$$\frac{\lambda + \mu_0 K H_1^2}{\lambda + \mu_0 K H_2^2} = \frac{\lambda + \mu_0 K H_4^2}{\lambda + \mu_0 K H_3^2}$$

Cross-multiplying and simplifying (as shown in the detailed derivation):

$$(\lambda + \mu_0 K H_1^2)(\lambda + \mu_0 K H_3^2) = (\lambda + \mu_0 K H_2^2)(\lambda + \mu_0 K H_4^2)$$

From the adiabatic conditions, we can also derive:

$$\lambda(T_h^2 - T_c^2) = \frac{\mu_0 V^2}{n^2 K} T_h^2 T_c^2 (M_1^2 - M_2^2)$$

$$\lambda(T_h^2 - T_c^2) = \frac{\mu_0 V^2}{n^2 K} T_h^2 T_c^2 (M_4^2 - M_3^2)$$

Equating these:

$$M_1^2 - M_2^2 = M_4^2 - M_3^2$$

$$M_1^2 + M_3^2 = M_2^2 + M_4^2$$

Final Analytical Expression:

$$M_1 = \sqrt{M_2^2 + M_4^2 - M_3^2}$$

T3-C3 [0.8 pts]
Derivation & Working:

Given data:

- $n = 2.0$ mol
- $K = 1.87 \times 10^{-6}$ K·m³/mol
- $H_1 = 411624$ A/m, $H_2 = 311306$ A/m, $H_3 = 204618$ A/m, $H_4 = 240446$ A/m
- Liquid helium: $V_{\text{He}} = 1.00$ L $= 1.00 \times 10^{-3}$ m³, $T_{\text{He, initial}} = 1.00$ K
- $c = 100$ J/(kg·K), $\rho = 130$ kg/m³
- $\mu_0 = 1.25663706127 \times 10^{-6}$ kg·m·A⁻²·s⁻²

Mass of helium:

$$m_{\text{He}} = \rho V_{\text{He}} = 130 \times 1.00 \times 10^{-3} = 0.13 \text{ kg}$$

Heat capacity of helium:

$$C_{\text{He}} = m_{\text{He}} c = 0.13 \times 100 = 13 \text{ J/K}$$

For the Carnot cycle, the heat absorbed from the cold reservoir during process $2 \to 3$:

$$Q_c = \frac{\mu_0 n K (H_2^2 - H_3^2)}{2T_c}$$

Assuming the cold reservoir temperature $T_c = T_{\text{He, initial}} = 1.00$ K:

$$Q_c = \frac{(1.25663706127 \times 10^{-6}) \times 2.0 \times (1.87 \times 10^{-6}) \times (311306^2 - 204618^2)}{2 \times 1.00}$$

$$Q_c = \frac{(1.25663706127 \times 10^{-6}) \times 2.0 \times (1.87 \times 10^{-6}) \times (5.5043 \times 10^{10})}{2}$$

$$Q_c = 0.129 \text{ J}$$

The temperature change of helium:

$$\Delta T_{\text{He}} = -\frac{Q_c}{C_{\text{He}}} = -\frac{0.129}{13} = -0.0099 \text{ K}$$

Final temperature:

$$T_{\text{He, final}} = T_{\text{He, initial}} + \Delta T_{\text{He}} = 1.00 + (-0.0099) = 0.99 \text{ K}$$

(Note: The answer is given to 2 significant figures, limited by the least-precise input $\rho = 130$ kg/m³.)

Final Numerical Value (with units):

"**0.99 K**"

T3-C4 [2.0 pts]
Derivation & Working:

For continuous cooling with constant power P, we consider infinitesimal Carnot cycles. The cooled body has heat capacity C_c and temperature T_c (which varies with time).

For each infinitesimal cycle:

- Heat removed from cooled body: $dQ_c = -C_c dT_c$
- From Carnot relation: $\frac{dQ_c}{dQ_h} = \frac{T_c}{T_h}$, so $dQ_h = dQ_c \frac{T_h}{T_c}$
- Work done: $dW = dQ_h - dQ_c = dQ_c\left(\frac{T_h}{T_c} - 1\right) = -C_c dT_c\left(\frac{T_h - T_c}{T_c}\right)$

Power is constant: $P = \frac{dW}{dt}$, so:

$$Pdt = -C_c \frac{T_h - T_c}{T_c} dT_c = -C_c\left(\frac{T_h}{T_c} - 1\right) dT_c$$

$$dt = -\frac{C_c}{P}\left(\frac{T_h}{T_c} - 1\right) dT_c$$

Integrating from $t = 0$ (when $T_c = T_0$) to time t (when $T_c = T$):

$$t = -\frac{C_c}{P} \int_{T_0}^{T} \left(\frac{T_h}{T_c} - 1\right) dT_c$$

$$t = -\frac{C_c}{P} [T_h \ln T_c - T_c]_{T_0}^{T}$$

$$t = -\frac{C_c}{P} [(T_h \ln T - T) - (T_h \ln T_0 - T_0)]$$

$$t = \frac{C_c}{P}\left[T_h \ln\left(\frac{T_0}{T}\right) + T - T_0\right]$$

Final Analytical Expression:

$$t = \frac{C_c}{P}\left[T_h \ln\left(\frac{T_0}{T}\right) + T - T_0\right]$$

T3-C5 [1.5 pts]
Derivation & Working:

The coefficient of performance is defined as $\text{COP} = \frac{Q_c}{W}$.

For the entire cooling process from T_0 to T:

- Total heat removed from cooled body: $Q_c = C_c(T_0 - T)$
- Total work done: $W = Pt = P \cdot \frac{C_c}{P}\left[T_h \ln\left(\frac{T_0}{T}\right) + T - T_0\right] = C_c\left[T_h \ln\left(\frac{T_0}{T}\right) + T - T_0\right]$

Therefore:

$$\text{COP} = \frac{C_c(T_0 - T)}{C_c\left[T_h \ln\left(\frac{T_0}{T}\right) + T - T_0\right]} = \frac{T_0 - T}{T_h \ln\left(\frac{T_0}{T}\right) - (T_0 - T)}$$

Final Analytical Expression:

$$\text{COP} = \frac{T_0 - T}{T_h \ln\left(\frac{T_0}{T}\right) - (T_0 - T)}$$