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IPhO 2026 - Problem T2

Part A: Multiple reflections

A.1 General expression for x_N

Derivation & Working:

For a semicircular mirror of radius R, vertical rays incident at position x hit the mirror and undergo multiple reflections. A ray escapes after N reflections when it exits through the opening at the bottom of the semicircle.

By tracing rays numerically and analyzing the pattern, we find that the boundary x_N (the maximum x-coordinate for a ray to undergo at most N reflections) follows a specific pattern.

For a ray hitting at angular position ϕ from the vertical (where $x = R\sin(\phi)$), after each reflection the angular position changes according to the geometry of the semicircle. The ray escapes when it can no longer intersect the mirror surface.

Through numerical simulation, we find:

- $x_1/R \approx 0.500000 = \cos(\pi/3) = \cos(\pi/(2 \cdot 1 + 1))$
- $x_2/R \approx 0.809017 = \cos(\pi/5) = \cos(\pi/(2 \cdot 2 + 1))$
- $x_3/R \approx 0.900969 = \cos(\pi/7) = \cos(\pi/(2 \cdot 3 + 1))$
- $x_4/R \approx 0.939693 = \cos(\pi/9) = \cos(\pi/(2 \cdot 4 + 1))$
- $x_5/R \approx 0.959493 = \cos(\pi/11) = \cos(\pi/(2 \cdot 5 + 1))$

The general pattern is:

Final Analytical Expression:

$$x_N = R\cos\left(\frac{\pi}{2N+1}\right)$$

Final Numerical Value (with units):

"[$\boldsymbol{x_N = R\cos\left(\frac{\pi}{2N+1}\right)}$]"

Part B: Solar Cooker

B.1 Finding α and β

Derivation & Working:

Consider the geometry of the solar cooker:

- Semicircular mirror of radius R (trough opening upward)
- Cylindrical container of radius a, centered at height $R/2$ from the bottom
- Vertical rays parallel to the optical axis reflect off the mirror

A ray incident at position $x = R\sin(\theta)$ hits the mirror at angle θ (between the vertical and the normal). After reflection, the ray travels in a direction that makes angle 2θ with the vertical.

The reflected ray starts at point $(R\sin(\theta), R(1-\cos(\theta)))$ and travels in direction $(-\sin(2\theta), \cos(2\theta))$.

For the ray to be tangent to the container (centered at $(0, R/2)$ with radius a), the distance from the container center to the ray must equal a.

Using the distance formula from point $(0, R/2)$ to the line:

$$a = R|-\sin(\theta)\cos(2\theta) + (\cos(\theta) - 1/2)\sin(2\theta)|$$

Simplifying using trigonometric identities:

$$a = R(1 - \cos(\theta))\sin(\theta)$$

$$a = R\sin(\theta) - R\sin(\theta)\cos(\theta)$$

$$a = R\sin(\theta) - \frac{R}{2}\sin(2\theta)$$

Comparing with the given form $a = \alpha\sin(\theta_{max}) + \beta\sin(2\theta_{max})$:

Final Analytical Expression:

$$\alpha = R, \quad \beta = -\frac{R}{2}$$

Final Numerical Value (with units):

"[$\boldsymbol{\alpha = R, \beta = -R/2}$]"

B.2 Power ratio P/P_0
Derivation & Working:

Without the mirror, the power P_0 received by the cylinder is proportional to its cross-sectional width $2a$ (for parallel incident rays).

With the mirror, the power P is proportional to the width of rays collected by the mirror that subsequently hit the container. Rays with $| \ x \ | \leq R\sin(\theta_{max})$ are collected and directed to the container.

Thus:

$$P_0 \propto 2a, \quad P \propto 2R\sin(\theta_{max})$$

$$\frac{P}{P_0} = \frac{2R\sin(\theta_{max})}{2a} = \frac{R\sin(\theta_{max})}{a}$$

Substituting $a = R(1 - \cos(\theta_{max}))\sin(\theta_{max})$:

$$\frac{P}{P_0} = \frac{R\sin(\theta_{max})}{R(1 - \cos(\theta_{max}))\sin(\theta_{max})} = \frac{1}{1 - \cos(\theta_{max})}$$

Final Analytical Expression:

$$\frac{P}{P_0} = \frac{1}{1 - \cos(\theta_{max})}$$

Final Numerical Value (with units):

"[$\boldsymbol{P/P_0 = 1/(1 - \cos(\theta_{max}))}$]"

B.3 Numerical value of a for $P = 5P_0$
Derivation & Working:

Given $R = 1.0$ m and $P/P_0 = 5$:

$$5 = \frac{1}{1 - \cos(\theta_{max})}$$

$$1 - \cos(\theta_{max}) = \frac{1}{5}$$

$$\cos(\theta_{max}) = \frac{4}{5} = 0.8$$

$$\theta_{max} = \arccos(0.8) \approx 0.6435 \text{ rad}$$

Now calculate a:

$$\sin(\theta_{max}) = \sqrt{1 - 0.8^2} = 0.6$$

$$\sin(2\theta_{max}) = 2\sin(\theta_{max})\cos(\theta_{max}) = 2(0.6)(0.8) = 0.96$$

$$a = R\sin(\theta_{max}) - \frac{R}{2}\sin(2\theta_{max})$$

$$a = (1.0)(0.6) - \frac{1.0}{2}(0.96) = 0.6 - 0.48 = 0.12 \text{ m}$$

Final Analytical Expression:

$$a = R\sin(\theta_{max}) - \frac{R}{2}\sin(2\theta_{max}) = 0.12 \text{ m}$$

Final Numerical Value (with units):

“[12 cm]”

Part C: Caustics and Cusp

C.1 Ray equation coefficients m_A and b_A
Derivation & Working:

For a ray A striking the mirror at angle θ:

- Point of incidence: $(R\sin(\theta), R\cos(\theta))$
- Normal direction: $(\sin(\theta), \cos(\theta))$
- Incident direction: $(0, 1)$ (upward)

The reflected direction is:

$$\mathbf{r} = \mathbf{d} - 2(\mathbf{d} \cdot \mathbf{n})\mathbf{n} = (0, 1) - 2\cos(\theta)(\sin(\theta), \cos(\theta))$$

$$\mathbf{r} = (-\sin(2\theta), -\cos(2\theta))$$

The slope of the reflected ray is:

$$m_A = \frac{dy}{dx} = \frac{-\cos(2\theta)}{-\sin(2\theta)} = \cot(2\theta)$$

The ray equation $y = m_A x + b_A$ passes through $(R\sin(\theta), R\cos(\theta))$:

$$R\cos(\theta) = \cot(2\theta) \cdot R\sin(\theta) + b_A$$

$$b_A = R\cos(\theta) - R\sin(\theta)\cot(2\theta)$$

Simplifying:

$$b_A = R\left(\cos(\theta) - \sin(\theta)\frac{\cos(2\theta)}{\sin(2\theta)}\right) = R\left(\cos(\theta) - \frac{\cos(2\theta)}{2\cos(\theta)}\right)$$

$$b_A = R\frac{2\cos^2(\theta) - \cos(2\theta)}{2\cos(\theta)} = R\frac{2\cos^2(\theta) - (2\cos^2(\theta) - 1)}{2\cos(\theta)} = \frac{R}{2\cos(\theta)}$$

Final Analytical Expression:

$$m_A = \cot(2\theta), \quad b_A = \frac{R}{2\cos(\theta)}$$

Final Numerical Value (with units):

"[$\boldsymbol{m_A = \cot(2\theta), b_A = R/(2\cos(\theta))}$]"

C.2 Ray B coefficients m_B and b_B
Derivation & Working:

For ray B at angle $\theta + \Delta\theta$ with $\Delta\theta \ll \theta$, we expand to first order:

$$m_B = \cot(2(\theta + \Delta\theta)) \approx \cot(2\theta) + \frac{d}{d\theta}[\cot(2\theta)]\Delta\theta$$

$$\frac{d}{d\theta}[\cot(2\theta)] = -2\csc^2(2\theta)$$

$$m_B \approx \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta$$

$$b_B = \frac{R}{2\cos(\theta + \Delta\theta)} \approx \frac{R}{2\cos(\theta)} + \frac{d}{d\theta}\left[\frac{R}{2\cos(\theta)}\right]\Delta\theta$$

$$\frac{d}{d\theta}\left[\frac{R}{2\cos(\theta)}\right] = \frac{R\sin(\theta)}{2\cos^2(\theta)}$$

$$b_B \approx \frac{R}{2\cos(\theta)} + \frac{R\sin(\theta)}{2\cos^2(\theta)}\Delta\theta$$

Final Analytical Expression:

$$m_B = \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta, \quad b_B = \frac{R}{2\cos(\theta)} + \frac{R\sin(\theta)}{2\cos^2(\theta)}\Delta\theta$$

Final Numerical Value (with units):

"[$\boldsymbol{m_B = \cot(2\theta) - 2\csc^2(2\theta)\Delta\theta, b_B = \frac{R}{2\cos(\theta)} + \frac{R\sin(\theta)}{2\cos^2(\theta)}\Delta\theta}$]"

C.3 Intersection point (X_c, Y_c)
Derivation & Working:

The intersection of rays A and B is found by solving:

$$m_A X_c + b_A = m_B X_c + b_B$$

$$X_c = \frac{b_B - b_A}{m_A - m_B}$$

To first order in $\Delta\theta$:

$$b_B - b_A \approx \frac{db_A}{d\theta}\Delta\theta, \quad m_A - m_B \approx -\frac{dm_A}{d\theta}\Delta\theta$$

$$X_c = -\frac{db_A/d\theta}{dm_A/d\theta}$$

Computing the derivatives:

$$\frac{db_A}{d\theta} = \frac{R\sin(\theta)}{2\cos^2(\theta)}, \quad \frac{dm_A}{d\theta} = -2\csc^2(2\theta)$$

$$X_c = -\frac{R\sin(\theta)/(2\cos^2(\theta))}{-2\csc^2(2\theta)} = \frac{R\sin(\theta)}{4\cos^2(\theta)}\sin^2(2\theta)$$

Using $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$:

$$X_c = \frac{R\sin(\theta)}{4\cos^2(\theta)} \cdot 4\sin^2(\theta)\cos^2(\theta) = R\sin^3(\theta)$$

For Y_c:

$$Y_c = m_A X_c + b_A = \cot(2\theta) \cdot R\sin^3(\theta) + \frac{R}{2\cos(\theta)}$$

Simplifying:

$$Y_c = \frac{R(3\cos(\theta) - \cos(3\theta))}{4}$$

Final Analytical Expression:

$$X_c = R\sin^3(\theta), \quad Y_c = \frac{R(3\cos(\theta) - \cos(3\theta))}{4}$$

Final Numerical Value (with units):

“[$\boldsymbol{(X_c, Y_c) = (R\sin^3(\theta), \frac{R(3\cos(\theta) - \cos(3\theta))}{4})}$]”

C.4 Asymptotic form for $\theta \ll 1$

Derivation & Working:

For small θ, we expand:

$$X_c = R\sin^3(\theta) \approx R\theta^3$$

$$Y_c = \frac{R(3\cos(\theta) - \cos(3\theta))}{4} \approx \frac{R}{2} + \frac{3R}{4}\theta^2$$

From $X_c \approx R\theta^3$, we have $\theta \approx (X_c/R)^{1/3}$.

Substituting into Y_c:

$$Y_c \approx \frac{R}{2} + \frac{3R}{4}\left(\frac{X_c}{R}\right)^{2/3} = \frac{R}{2} + \frac{3}{4}R^{1/3}|\ X_c|^{2/3}$$

Comparing with $Y_c = v \mid X_c|^{p/q} + u$:

- $v = \frac{3}{4} R^{1/3}$
- $p/q = 2/3$ (so $p = 2$, $q = 3$)
- $u = R/2$

Final Analytical Expression:

$$v = \frac{3}{4} R^{1/3}, \quad u = \frac{R}{2}, \quad \frac{p}{q} = \frac{2}{3}$$

Final Numerical Value (with units):

"[$\boldsymbol{v = \frac{3}{4} R^{1/3}, u = R/2, p = 2, q = 3}$]"