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IPhO 2026 - Problem T1

Part A: Hydrostatic Gate

T1-A1: Calculate the value of a
Derivation & Working:

The cubic block of side length a is rotated 45° and fixed at point O on the partition. The block can rotate about an axis perpendicular to the page through O.

For equilibrium at the maximum water level difference Δh, the net torque about O must be zero.

Setting up coordinates with O at the origin and the partition as the y-axis, the square vertices are:

- Top: $(\sqrt{2}a/4, \sqrt{2}a/4)$
- Right: $(3\sqrt{2}a/4, -\sqrt{2}a/4)$
- Bottom: $(\sqrt{2}a/4, -3\sqrt{2}a/4)$
- Left: $(-\sqrt{2}a/4, -\sqrt{2}a/4)$

The center of mass is at $(\sqrt{2}a/4, -\sqrt{2}a/4)$.

Integrating the hydrostatic pressure over each edge segment (accounting for the partition at $x = 0$ separating water levels h_L and $h_R = h_L - \Delta h$), the total hydrostatic torque about O is:

$$\tau_{\text{hydro}} = \frac{a^3 g\rho_0(\Delta h + \sqrt{2}a)}{4}$$

The weight of the block is $W = 3\rho_0 a^3 g$, acting downward at the center of mass. The weight torque is:

$$\tau_{\text{weight}} = -x_{\text{cm}}W = -\frac{\sqrt{2}a}{4} \cdot 3\rho_0 a^3 g = -\frac{3\sqrt{2}\rho_0 g a^4}{4}$$

For equilibrium, $\tau_{\text{total}} = \tau_{\text{hydro}} + \tau_{\text{weight}} = 0$:

$$\frac{a^3 g\rho_0(\Delta h + \sqrt{2}a)}{4} - \frac{3\sqrt{2}\rho_0 g a^4}{4} = 0$$

Dividing by $a^3 g\rho_0/4$:

$$\Delta h + \sqrt{2}a - 3\sqrt{2}a = 0$$

$$\Delta h = 2\sqrt{2}a$$

Solving for a:

$$a = \frac{\Delta h}{2\sqrt{2}} = \frac{\sqrt{2}\Delta h}{4}$$

Final Analytical Expression:

$$a = \frac{\sqrt{2}\Delta h}{4}$$

Final Numerical Value (with units):

$$a = \frac{\sqrt{2} \times 1.41\ \mathrm{m}}{4} = 0.499\ \mathrm{m}$$

Part B: Electron-Positron Pair

T1-B1: Maximum separation distance for $\mu = 4$
Derivation & Working:

For a two-body system with equal masses m and charges $\pm e$:

- Each particle has angular momentum $\mu\hbar$ about the center of mass
- Total angular momentum: $L = 2\mu\hbar = 8\hbar$ (for $\mu = 4$)
- Initial separation: $r_0 = 100a_0$
- Reduced mass: $\mu_{\mathrm{red}} = m/2$

Each particle has $L_{\mathrm{each}} = \mu\hbar = m(r_0/2)v$, so $v = 2\mu\hbar/(mr_0)$.

Total kinetic energy:

$$KE = \frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2 = m\left(\frac{2\mu\hbar}{mr_0}\right)^2 = \frac{4\mu^2\hbar^2}{mr_0^2}$$

Potential energy:

$$PE = -\frac{ke^2}{r_0} = -\frac{\hbar^2}{ma_0r_0}$$

Total energy:

$$E = \frac{4\mu^2\hbar^2}{mr_0^2} - \frac{\hbar^2}{ma_0r_0} = \frac{\hbar^2}{ma_0^2}\left(\frac{4\mu^2a_0^2}{r_0^2} - \frac{a_0}{r_0}\right)$$

For $\mu = 4$ and $r_0 = 100a_0$:

$$E = \frac{\hbar^2}{ma_0^2}\left(\frac{64}{10000} - \frac{1}{100}\right) = \frac{\hbar^2}{ma_0^2}(-0.0036) < 0$$

The system is bound. The turning points satisfy:

$$E = \frac{L^2}{2\mu_{\mathrm{red}}r^2} - \frac{ke^2}{r}$$

With $L = 2\mu\hbar = 8\hbar$ and $\mu_{\mathrm{red}} = m/2$:

$$E = \frac{64\hbar^2}{mr^2} - \frac{\hbar^2}{ma_0r}$$

Multiplying by $mr^2/\hbar^2$:

$$\frac{Emr^2}{\hbar^2} = 64 - \frac{r}{a_0}$$

Substituting $E = -0.0036\hbar^2/(ma_0^2)$:

$$-0.0036\frac{r^2}{a_0^2} = 64 - \frac{r}{a_0}$$

Let $x = r/a_0$:

$$-0.0036x^2 = 64 - x$$

$$0.0036x^2 - x + 64 = 0$$

Solving:

$$x = \frac{1 \pm \sqrt{1 - 4(0.0036)(64)}}{2(0.0036)} = \frac{1 \pm \sqrt{1 - 0.9216}}{0.0072} = \frac{1 \pm 0.28}{0.0072}$$

$$x_1 = \frac{1 - 0.28}{0.0072} = 100, \quad x_2 = \frac{1 + 0.28}{0.0072} = \frac{1.28}{0.0072} = \frac{1600}{9}$$

Final Analytical Expression:

$$r_{\text{max}} = \frac{1600}{9}a_0$$

Final Numerical Value:

$$r_{\text{max}} = \frac{1600}{9}a_0 \approx 178a_0$$

T1-B2: Scattering angle for $\mu = 15/2$
Derivation & Working:

For $\mu = 15/2$:

$$L = 2\mu\hbar = 15\hbar$$

Energy:

$$E = \frac{4\mu^2\hbar^2}{mr_0^2} - \frac{\hbar^2}{ma_0r_0} = \frac{\hbar^2}{ma_0^2}\left(\frac{4(225/4)}{10000} - \frac{1}{100}\right) = \frac{\hbar^2}{ma_0^2}\left(\frac{225}{10000} - \frac{1}{100}\right) = \frac{\hbar^2}{ma_0^2}(0.0125) > 0$$

The system is unbound (hyperbolic trajectory).

Step 1: Calculate eccentricity using Hint 1

$$\varepsilon = \sqrt{1 + \frac{4L^2E}{k^2e^4m}}$$

Using $ke^2 = \hbar^2/(ma_0)$, so $k^2e^4 = \hbar^4/(m^2a_0^2)$:

$$\frac{4L^2E}{k^2e^4m} = \frac{4(225\hbar^2)(0.0125\hbar^2/(ma_0^2))}{(\hbar^4/(m^2a_0^2))m} = 4 \cdot 225 \cdot 0.0125 = 11.25 = \frac{45}{4}$$

$$\varepsilon = \sqrt{1 + \frac{45}{4}} = \sqrt{\frac{49}{4}} = \frac{7}{2}$$

Step 2: Derive scattering angle from Hint 2

From Hint 2, the conic equation is:

$$r = \frac{a}{1 - \varepsilon \cos\theta}$$

For a hyperbola ($\varepsilon > 1$), the asymptotes occur at $r \to \infty$, which requires:

$$1 - \varepsilon \cos\theta_\infty = 0 \Rightarrow \cos\theta_\infty = \frac{1}{\varepsilon} = \frac{2}{7}$$

The angle between the two asymptotes is $2\theta_\infty$ where $\theta_\infty = \arccos(2/7)$.

The scattering angle χ (deflection from straight-line motion) is:

$$\chi = \pi - 2\theta_\infty = \pi - 2\arccos\left(\frac{2}{7}\right)$$

Using the identity $\arccos(x) = \frac{\pi}{2} - \arcsin(x)$:

$$\chi = \pi - 2\left(\frac{\pi}{2} - \arcsin\left(\frac{2}{7}\right)\right) = 2\arcsin\left(\frac{2}{7}\right)$$

Step 3: Verify the initial point is at periapsis

At the initial point, the velocity is perpendicular to the separation vector ($r_0 = 100a_0$). For a central force orbit, $dr/dt = 0$ when velocity is perpendicular to position, which occurs at turning points (periapsis or apoapsis).

For a hyperbola, there is only one turning point (periapsis). We verify that r_0 equals the periapsis distance.

The orbit equation parameter (semi-latus rectum) is:

$$p = \frac{L^2}{\mu_{\text{red}} k e^2} = \frac{225\hbar^2}{(m/2)(\hbar^2/(ma_0))} = 450a_0$$

The periapsis distance for a hyperbola is:

$$r_p = \frac{p}{1+\varepsilon} = \frac{450a_0}{1+7/2} = \frac{450a_0}{9/2} = 100a_0$$

This confirms $r_0 = 100a_0$ IS the periapsis distance.

Step 4: Calculate the angle

At periapsis, the velocity is perpendicular to the position vector (given). As $r \to \infty$, the velocity approaches $\vec{u}_\infty$ along the asymptote direction.

The angle between the periapsis direction and the asymptote is $\theta_\infty = \arccos(1/\varepsilon) = \arccos(2/7)$.

At periapsis, the velocity is at angle $\pi/2$ from the radius vector. At infinity, the velocity is parallel to the asymptote (at angle θ_∞ from the axis).

The angle between initial velocity (at $\pi/2$) and final velocity (at angle $\theta_\infty - \pi/2$ relative to the perpendicular) is:

$$\text{Angle} = \frac{\pi}{2} - \theta_\infty + \frac{\pi}{2} - \frac{\pi}{2} = \frac{\pi}{2} - \arccos\left(\frac{2}{7}\right) = \arcsin\left(\frac{2}{7}\right)$$

Final Analytical Expression:

$$\text{Angle} = \arcsin\left(\frac{2}{7}\right)$$

Final Numerical Value (with units):

$$\text{Angle} = \arcsin\left(\frac{2}{7}\right) \approx 16.6^\circ$$

Part C: Photodissociation of Ozone

T1-C1: Minimum angular frequency ω_{min}
Derivation & Working:

Conservation of momentum (photon along x-axis, O_2 at angle θ):

$$\frac{\hbar\omega}{c} = p_2 \cos\theta + p_1 \cos\phi$$

$$0 = p_2 \sin\theta - p_1 \sin\phi$$

where p_2 is the O_2 momentum, p_1 is the O momentum, and ϕ is the O angle.

Conservation of energy:

$$\hbar\omega = \Delta U + \frac{p_2^2}{2(2m)} + \frac{p_1^2}{2m} = \Delta U + \frac{p_2^2}{4m} + \frac{p_1^2}{2m}$$

From the momentum equations:

$$p_1 \sin\phi = p_2 \sin\theta$$

$$p_1 \cos\phi = \frac{\hbar\omega}{c} - p_2 \cos\theta$$

Squaring and adding:

$$p_1^2 = p_2^2 \sin^2\theta + \left(\frac{\hbar\omega}{c} - p_2\cos\theta\right)^2 = p_2^2 + \left(\frac{\hbar\omega}{c}\right)^2 - 2\frac{\hbar\omega}{c} p_2 \cos\theta$$

Substituting into the energy equation:

$$\hbar\omega = \Delta U + \frac{p_2^2}{4m} + \frac{1}{2m}\left[p_2^2 + \left(\frac{\hbar\omega}{c}\right)^2 - 2\frac{\hbar\omega}{c}p_2\cos\theta\right]$$

$$\hbar\omega = \Delta U + \frac{3p_2^2}{4m} + \frac{(\hbar\omega)^2}{2mc^2} - \frac{\hbar\omega}{mc}p_2\cos\theta$$

Rearranging as a quadratic in p_2:

$$\frac{3}{4m}p_2^2 - \frac{\hbar\omega\cos\theta}{mc}p_2 + \left[\frac{(\hbar\omega)^2}{2mc^2} + \Delta U - \hbar\omega\right] = 0$$

For the minimum ω (threshold), the discriminant must be zero:

$$\left(\frac{\hbar\omega\cos\theta}{mc}\right)^2 - 4\left(\frac{3}{4m}\right)\left[\frac{(\hbar\omega)^2}{2mc^2} + \Delta U - \hbar\omega\right] = 0$$

$$\frac{(\hbar\omega)^2\cos^2\theta}{m^2c^2} - \frac{3}{m}\left[\frac{(\hbar\omega)^2}{2mc^2} + \Delta U - \hbar\omega\right] = 0$$

Multiplying by m^2c^2:

$$(\hbar\omega)^2\cos^2\theta - 3mc^2\left[\frac{(\hbar\omega)^2}{2mc^2} + \Delta U - \hbar\omega\right] = 0$$

$$(\hbar\omega)^2\cos^2\theta - \frac{3}{2}(\hbar\omega)^2 - 3mc^2\Delta U + 3mc^2\hbar\omega = 0$$

$$(\hbar\omega)^2\left(\cos^2\theta - \frac{3}{2}\right) + 3mc^2\hbar\omega - 3mc^2\Delta U = 0$$

Using $\cos^2\theta - \frac{3}{2} = \frac{1}{2}(\cos 2\theta - 2)$:

$$\frac{1}{2}(\cos 2\theta - 2)(\hbar\omega)^2 + 3mc^2\hbar\omega - 3mc^2\Delta U = 0$$

Multiplying by 2:

$$(\cos 2\theta - 2)(\hbar\omega)^2 + 6mc^2\hbar\omega - 6mc^2\Delta U = 0$$

Solving for $\hbar\omega$ using the quadratic formula with $A = \cos 2\theta - 2$, $B = 6mc^2$, $C = -6mc^2\Delta U$:

$$\hbar\omega = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-6mc^2 \pm \sqrt{36m^2c^4 - 4(\cos 2\theta - 2)(-6mc^2\Delta U)}}{2(\cos 2\theta - 2)}$$

The discriminant simplifies to:

$$B^2 - 4AC = 36m^2c^4 + 24mc^2\Delta U(\cos 2\theta - 2) = 36m^2c^4 - 24mc^2\Delta U(2 - \cos 2\theta)$$

For the minimum ω (smaller root, noting that $A = \cos 2\theta - 2 < 0$):

$$\hbar\omega_{\text{min}} = \frac{-6mc^2 + \sqrt{36m^2c^4 - 24mc^2\Delta U(2 - \cos 2\theta)}}{2(\cos 2\theta - 2)}$$

Since $\cos 2\theta - 2 = -(2 - \cos 2\theta)$, we rewrite:

$$\hbar\omega_{\text{min}} = \frac{6mc^2 - \sqrt{36m^2c^4 - 24mc^2\Delta U(2 - \cos 2\theta)}}{2(2 - \cos 2\theta)}$$

Factoring out $36m^2c^4$ from the square root:

$$\hbar\omega_{\text{min}} = \frac{6mc^2 - 6mc^2\sqrt{1 - \frac{2\Delta U(2-\cos 2\theta)}{3mc^2}}}{2(2 - \cos 2\theta)}$$

Final Analytical Expression:

$$\omega_{\text{min}} = \frac{3mc^2\left[1 - \sqrt{1 - \frac{2\Delta U(2-\cos 2\theta)}{3mc^2}}\right]}{\hbar(2 - \cos 2\theta)}$$

T1-C2: Calculate $\hbar\omega_{\text{min}} - \Delta U$ for $\theta = \pi/6$
Derivation & Working:

For $\theta = \pi/6$:

$$\cos 2\theta = \cos(\pi/3) = \frac{1}{2}$$

$$2 - \cos 2\theta = \frac{3}{2}$$

Let $E = \hbar\omega_{\text{min}}$. The quadratic becomes:

$$-\frac{3}{2}E^2 + 6mc^2 E - 6mc^2 \Delta U = 0$$

Multiplying by $-2/3$:

$$E^2 - 4mc^2 E + 4mc^2 \Delta U = 0$$

Solutions:

$$E = \frac{4mc^2 \pm \sqrt{16m^2c^4 - 16mc^2\Delta U}}{2} = 2mc^2 \pm 2mc^2\sqrt{1 - \frac{\Delta U}{mc^2}}$$

For the smaller root (physical, $E \approx \Delta U$):

$$E = 2mc^2\left(1 - \sqrt{1 - \frac{\Delta U}{mc^2}}\right)$$

Using the Taylor expansion $(1-x)^{1/2} \approx 1 - x/2 - x^2/8$ for small $x = \Delta U/(mc^2)$:

$$E \approx 2mc^2\left(1 - 1 + \frac{\Delta U}{2mc^2} + \frac{\Delta U^2}{8m^2c^4}\right) = \Delta U + \frac{\Delta U^2}{4mc^2}$$

Thus:

$$\hbar\omega_{\text{min}} - \Delta U \approx \frac{\Delta U^2}{4mc^2}$$

Substituting values:

- $\Delta U = 1.10$ eV
- $m = 16.0$ amu $= 16.0 \times 9.3149410372 \times 10^8$ eV/c^2
- $mc^2 = 1.490 \times 10^{10}$ eV

$$\hbar\omega_{\text{min}} - \Delta U = \frac{(1.10 \text{ eV})^2}{4 \times 1.490 \times 10^{10} \text{ eV}} = 2.03 \times 10^{-11} \text{ eV}$$

Final Numerical Value (with units):

$$\hbar\omega_{\text{min}} - \Delta U = 2.03 \times 10^{-11} \text{ eV}$$